Tuesday, November 17, 2015

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Problem 28

Problem. Find the sum of the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$.

Solution. Use partial fractions.

$$\begin{aligned} \frac{1}{(2n+1)(2n+3)} &= \frac{A}{2n+1} + \frac{B}{2n+3}, \\ 1 &= A(2n+3) + B(2n+1). \end{aligned}$$

By substituting $n = -\frac{1}{2}$ and $n = -\frac{3}{2}$, we get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right).$$

Write out the first few terms:

$$\begin{aligned} a_1 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right), \\ a_2 &= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right), \\ a_3 &= \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right), \\ a_4 &= \frac{1}{2}\left(\frac{1}{9} - \frac{1}{11}\right), \\ a_5 &= \frac{1}{2}\left(\frac{1}{11} - \frac{1}{13}\right), \end{aligned}$$

With the cancelation, we see that the partial sums are

$$\begin{aligned}
S_1 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right),\\
S_2 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right),\\
S_3 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{9}\right),\\
S_4 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{11}\right),\\
S_5 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{13}\right),
\end{aligned}$$

The pattern is clear. $S_n = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2n+3}\right)$. Therefore, the series converges to $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2n+3}\right) = \frac{1}{6}$.

Problem 34

Problem. Find the sum of the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2}$.

Solution. The denominator factors as $9n^2 + 3n - 2 = (3n-1)(3n+2)$. The partial fraction decomposition gives us

$$\frac{1}{9n^2+3n-2} = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right).$$

Write out the first few terms:

$$
\begin{aligned}
a_1 &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right), \\
a_2 &= \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right), \\
a_3 &= \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right), \\
a_4 &= \frac{1}{3}\left(\frac{1}{11} - \frac{1}{14}\right), \\
a_5 &= \frac{1}{3}\left(\frac{1}{14} - \frac{1}{17}\right),
\end{aligned}
$$

With the cancelation, we see that the partial sums are

$$
\begin{aligned}
S_1 &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right), \\
S_2 &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{8}\right), \\
S_3 &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{11}\right), \\
S_4 &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{14}\right), \\
S_5 &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{17}\right),
\end{aligned}
$$

The pattern is clear. $S_n = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right)$. Therefore, the series converges to $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{6}$.

Problem 41

Problem. Determine the convergence or divergence of the series $\sum_{n=0}^{\infty} (1.075)^n$.

Solution. This is a geometric series with ratio $r = 1.075 > 1$. Therefore, it diverges.

Problem 42

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{1000}$.

Solution. This is a geometric series with ratio $r = 3 > 1$. Therefore, it diverges.

Problem 44

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$.

Solution. This may be a telescoping series. Let's see. Write out the first few terms and the first few partial sums.

$$\begin{aligned}
a_1 &= \frac{1}{1} - \frac{1}{3}, \\
a_2 &= \frac{1}{2} - \frac{1}{4}, \\
a_3 &= \frac{1}{3} - \frac{1}{5}, \\
a_4 &= \frac{1}{4} - \frac{1}{6}, \\
a_5 &= \frac{1}{5} - \frac{1}{7},
\end{aligned}$$

Then the partial sums are

$$\begin{aligned}
S_1 &= \frac{1}{1} - \frac{1}{3}, \\
S_2 &= \frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}, \\
S_3 &= \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}, \\
S_4 &= \frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}, \\
S_5 &= \frac{1}{1} + \frac{1}{2} - \frac{1}{6} - \frac{1}{7},
\end{aligned}$$

The pattern is clear. $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$. Therefore, the series converges to $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}$.

Problem 45

Problem. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$.

Solution. This may be a telescoping series. Let's see. Write out the first few terms and the first few partial sums.

$$\begin{aligned}
a_1 &= \frac{1}{2}-\frac{1}{3},\\
a_2 &= \frac{1}{3}-\frac{1}{4},\\
a_3 &= \frac{1}{4}-\frac{1}{5},\\
a_4 &= \frac{1}{5}-\frac{1}{6},\\
a_5 &= \frac{1}{6}-\frac{1}{7},
\end{aligned}$$

Then the partial sums are

$$\begin{aligned}
S_1 &= \frac{1}{2}-\frac{1}{3},\\
S_2 &= \frac{1}{2}-\frac{1}{4},\\
S_3 &= \frac{1}{2}-\frac{1}{5},\\
S_4 &= \frac{1}{2}-\frac{1}{6},\\
S_5 &= \frac{1}{2}-\frac{1}{7},
\end{aligned}$$

The pattern is clear. $S_n = \frac{1}{2}-\frac{1}{n+2}$. Therefore, the series converges to

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) = \frac{1}{2}.$$

Problem 47

Problem. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$.

Solution. Try the Divergence Test. And use L'Hôpital's Rule.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3^n}{n^3} \\ &= \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2} \\ &= \lim_{n\to\infty} \frac{3^n (\ln 3)^2}{6n} \\ &= \lim_{n\to\infty} \frac{3^n (\ln 3)^3}{6} \\ &= \infty.\end{aligned}$$

Because $a_n \nrightarrow 0$, the series diverges.

Problem 48

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5^n}$.

Solution. This is a geometric series with ratio $r = \frac{1}{5} < 1$, so it converges.

Problem 52

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} e^{-n}$.

Solution. This is a geometric series with ratio $r = e^{-1} < 1$, so it converges.

Problem 67

Problem. (a) Find the common ratio of the geometric series

$$1 + x + x^2 + x^3 + \cdots .$$

(b) Write the function that gives the sum of the series.

Solution. Divide any two consecutive terms to get the common ratio. Why not divide the second term by the first term? We get $r = \frac{x}{1} = x$. The first term is 1, so the sum of the series is

$$S = \frac{a}{1-r} = \frac{1}{1-x},$$

provided $|x| < 1$.

Problem 68

Problem. (a) Find the common ratio of the geometric series

$$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots .$$

(b) Write the function that gives the sum of the series.

Solution. Divide any two consecutive terms to get the common ratio. So let's divide the second term by the first term? We get $r = \frac{-\frac{x}{2}}{1} = -\frac{x}{2}$. The first term is 1, so the sum of the series is

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{1}{1-\left(-\frac{x}{2}\right)} \\ &= \frac{2}{2+x}, \end{aligned}$$

provided $\left|-\frac{x}{2}\right| < 1$, which is the same as $|x| < 2$.